The Nonsmooth optimal control problem for differential inclusion with parameter

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Abstract. In this paper we consider a linear controllable differential inclusion with parameter and under conditions of incomplete initial data. For this model of dynamic system the control problem by nonsmooth terminal functional is researched. The necessary and sufficient conditions for optimality are obtained.

Keywords: differential inclusion, control problem, parameter of system, ensemble of trajectories, nonsmooth functional, conditions of optimality.

1. Introduction.

Extreme problems, i.e. tasks for maximum and minimum of functionals, arise as optimization models of various applied problems from natural science, economics and technology. Mathematical optimization theory has been developed for such problems. In applied research, such sections of optimization theory as mathematical programming, calculus of variations, mathematical theory of optimal control and theory of optimal decision making have the widest applications [1, 2, 6, 10,15].

Mathematical modeling of various problems of economics and technology, such as making the best decision in economic planning and organization of production, when designing technical devices and managing technological processes lead to special optimization problems with non-smooth target functions. As a result of studies of optimization problems related to the control of complex systems and decision-making, methods of non-smooth optimization have been developed, sections of non-smooth and multivalued analysis have been formed [3,4,8, 15].

One of the approaches to non-smooth optimization problems is the minimax principle [7]. This approach is used when making a decision in conditions of incomplete information about the initial data and external influences [9]. In the case when the information about inaccurate parameters is minimal, i.e. only the permissible area of their change is known, according to this principle, the goal is to obtain a guaranteed value of the control quality criterion. And this is usually expressed as an optimization problem of a non-smooth function of the maximum or minimum type.

Each non-smooth function resulting from the maximization or minimization of the functional by a certain parameter has a specificity associated with the task of the functional itself and restrictions on the parameters. Therefore, the effectiveness of methods for solving the non-smooth optimization problem significantly depends on the properties of the objective functions and constraints on the system parameters [3,4].

Models described by differential inclusions with parameters are of great interest in the research of control systems under information constraints. Problems of optimal control of ensembles of trajectories are studied for such models [5,11,12]. In many cases, these tasks have not smooth optimization criteria.

In this paper, we consider a dynamic control system described by a linear differential inclusion with a parameter. Information about the initial state of the system is limited only by a known set of possible values. A terminal functional of the minimum function type is considered as a criterion for assessing the quality of management. For this model, the minimax problem of controlling an ensemble of trajectories is studied. The problem under consideration is investigated by methods of dynamic control systems, convex and multivalued analysis [13,14]. The results obtained is develop the results of the work [16,17,18]

2. Object of research and methods.

Consider a differential inclusion with parameters of the form

$$\dot{x} \in A(t,y)x + b(t,u,y), t \in T = [t_0, t_1], \ \ x(t_0) \in D, \ u \in V(y), \ \ y \in Y, \qquad (1)$$

where $x - n$ is a state vector, $u - m$ is a control vector, y - k -dimensional parameter, $A(t,y)$ - $n \times n$ - matrix, $b(t,u,y)$–multi-valued map. The peculiarity of this control system is that the information about the initial state of the system is inaccurate, i.e. only a convex compact set of possible initial states $D \subset R^n$ is known. In addition, parameter $y \in Y$ is involved in the control process, the value of which remains constant in the considered time interval $T = [t_0, t_1]$. The control domain is a convex compact subset $V(y)$ of space R^m, continuously dependent on parameter $y \in Y$. The set Y will also be considered a compact subset of the space R^k .

With respect to the right side of differential inclusion (1), we will assume that the following conditions are met:

1) the elements of matrix $A(t,y)$ are summable by $t \in T$ and continuous by $y \in Y$, with $\|A(t,y)\| \le \alpha(t)$, $\alpha(\cdot) \in L_1(T)$;

2) the multi-valued map $(t,u) \to b(t,u,y)$ is measurable by $t \in T$ and continuous by $(u,y) \in V \times Y$, and $\|b(t,u,y)\| \le \beta(t)$, $\beta(\cdot) \in L_1(T)$.

Definition 1. By permissible controls for system (1), we mean every measurable bounded m-vector function $u = u(t)$, $t \in T$, taking almost everywhere T values out of $V(y)$ for some $y \in Y$.

Definition 2. An admissible trajectory corresponding to control $u = u(t)$, $t \in T = [t_0, t_1]$, and parameter $y \in Y$ is an absolutely continuous n-vector function $x(t) = x(t, u, y)$ that satisfies almost everywhere on $T = [t_0, t_1]$ the differential inclusion (1) and the initial condition $x(t_0) \in D$.

Let: $U_T(y)$ be the set of permissible controls $u = u(\cdot)$, such that $u(t) \in V(y)$, $t \in T$; $H_T(u, y)$ be the set of all absolutely continuous solutions $x = x(t, u, y)$ of differential inclusion (1) with an initial condition $x(t_0) \in D$ for a given permissible control $u \in U_T(y)$ and parameter $y \in Y$. Under given conditions, $H_T(u, y)$ is a convex compact set in the space of continuous n-vector functions $C^n(T)$ [11].

Let the control quality of the dynamic system (1) be evaluated by a non-smooth terminal functional

$$J(x(\cdot), y) = \sum_{i=1}^{l} \inf_{z \in Z_i} (P_i(y)x(t_1), z), \tag{2}$$

where $P_i(y)$ - $s \times n$-matrix continuously dependent on parameter $y \in Y$, Z_i – is a closed bounded set of R^s. Since the initial state of the system (1) is set inaccurately, we will assume that the goal of control is to achieve a guaranteed value of the quality criterion $J(x(\cdot), y)$ of type (2), i.e. we will minimize the functionality of

$$G(u(\cdot), y) = \max_{x(\cdot) \in H_T(u, y)} J(x(\cdot), y).$$

In other words, for control system (1), consider the following minimax problem:

$$\max_{x(\cdot) \in H_T(u, y)} J(x(\cdot), y) \to \min, \; u \in U_T(y), y \in Y. \tag{3}$$

We will study the necessary and sufficient optimality conditions for the minimax problem (3).

Consider a set consisting of the ends of all trajectories $x(\cdot) \in H_T(u, y)$ at time $t_1 > t_0$:

$$X_T(t_1, u, y) = \{\xi \in R^n \mid \xi = x(t_1), x(\cdot) \in H_T(u, y)\}.$$

Due to the results of [11], $X_T(t_1, u, y)$ is a convex compact of R^n.

Let $F_y(t, \tau)$ be the fundamental matrix of solutions of equation $\dot{x} = A(t, y)x$, i.e.

$$\frac{\partial F_y(t, \tau)}{\partial t} = A(t, y)F_y(t, \tau), \; t \in T, \tau \in T, \; F_y(\tau, \tau) = E, \; E \text{ – single } n \times n \text{ - matrix.}$$

Using the Cauchy formula [1] for an absolutely continuous solution of differential inclusion (1), it is easy to make sure that the set $X_T(t_1,u,y)$ has the following representation[11]:

$$X_T(t_1,u,y) = F_y(t_1,t_0)D + \int_{t_0}^{t_1} F_y(t_1,\tau)b(\tau,u(\tau),y)d\tau \,. \tag{4}$$

Consider the function $\psi(t,y,q) = F_y'(t_1,t)q$. Given the formula (4), the support function $C(X_T(t_1,u,y),q) = \sup\limits_{\xi \in X_T(t_1,u,y)} (\xi,q)$ of the set $X(t_1,u,y)$ can be written as follows:

$$C(X_T(t_1,u,y),q) = C(D,\psi(t_0,y,q) + \int_{t_0}^{t_1} C(b(t,u(t),y),\psi(t,y,q)dt \,. \tag{5}$$

Let 's put: $Q(y) = \sum\limits_{i=1}^{l} P_i'(y) coZ_i$, where coZ_i is the convex hull of the set Z_i .

We have:

$$\sup_{x(\cdot)\in H_T(u,y)} J(x(\cdot),y) = \sup_{\xi \in X_T(t_1,u,y)} \sum_{i=1}^{l} \inf_{z\in Z_i} (P_i(y)\xi, z),$$

$$\sum_{i=1}^{l} \inf_{z\in Z_i} (P_i(y)\xi, z) = \inf_{z_i\in Z_i, i=\overline{1,l}} \left(\xi, \sum_{i=1}^{l} P_i'(y) z_i\right) = \inf_{z_i\in coZ_i, i=\overline{1,l}} \left(\xi, \sum_{i=1}^{l} P_i'(y) z_i\right).$$

Now, using the minimax theorem known from convex analysis [13], we obtain

$$\sup_{\xi\in X_T(u,y)} \sum_{i=1}^{l} \inf_{z\in Z_i} (P_i(y)\xi, z) = \inf_{z_i\in coZ_i, i=\overline{1,l}} \sup_{\xi\in X_T(t_1,u,y)} \left(\xi, \sum_{i=1}^{l} P_i'(y) z_i\right).$$

Therefore,

$$G(u(\cdot),y) = \max_{x(\cdot)\in H_T(u,y)} J(x(\cdot),y) = \inf_{q\in Q(y)} \sup_{\xi\in X_T(t_1,u,y)} (\xi,q) = \inf_{q\in Q(y)} C(X_T(t_1,u,y),q),$$

i.e. functional $G(u(\cdot),y) = \max\limits_{x(\cdot)\in H_T(u,y)} J(x(\cdot),y)$ has the following representation:

$$G(u(\cdot),y) = \inf_{q\in Q(u)} C(X_T(t_1,u,y),q) \,. \tag{6}$$

According to formula (6), the minimax problem (3) can be written as follows:

$$\inf_{q\in Q(y)} C(X_T(t_1,u,y),q)) \to \min,\ u(\cdot)\in U_T(y),\ y\in Y \,. \tag{7}$$

Thus, the minimax problem (3) is reduced to the problem of repeated minimization of the form (7). From the view of this problem, it is clear that it is a problem of controlling the terminal state of the $X_T(t_1,u,y)$ ensemble of trajectories of the dynamical system (1).

3.Main results.

We consider graphs of multivalued mappings $y \to U_T(y), y\in Y$ and $y \to Q(y), y\in Y$:

$$\Gamma_{U_T} = \{(y,u) : y \in Y, u \in U_T(y)\},\ \Gamma_Q = \{(y,q) : y \in Y, q \in Q(y)\}.$$

From the definition of the set of permissible controls $U_T(y)$, the set $Q(y)$ and the conditions imposed on the control area $V(y)$ and the sets $Z_i, i = \overline{1,l}$, it easily follows that the values of the multivalued maps $y \to U_T(y), y \in Y$ and $y \to Q(y), y \in Y$ are convex, closed and bounded subsets of spaces $L_2^m(T)$ and R^n, respectively. In addition, они непрерывны на компакте $Y \subset R^k$. Therefore, according to the results of the theory of multivalued maps [14], the following statements are true: the multivalued maps $y \to U_T(y), y \in Y$ and $y \to Q(y), y \in Y$ are closed and bounded, i.e. their graphs are closed and bounded sets in spaces $R^k \times L_2^m(T)$ and $R^k \times R^n$, respectively.

Let 's introduce the functionals:

$$\mu(u,y) = \inf_{q \in Q(y)} [C(D, \psi(t_0, y, q)) + \int_{t_0}^{t_1} C(b(t, u(t), y), \psi(t, y, q)) dt], u = u(\cdot) \in U_T(y), y \in Y,$$

(8)

$$\gamma(y,q) = C(D, \psi(t_0, y, q)) + \int_{t_0}^{t_1} \inf_{v \in V(y)} C(b(t,v,y), \psi(t,y,q)) dt,\ y \in Y, q \in Q(y). \tag{9}$$

Theorem 1. Functionals $\mu(u,y)$ and $\gamma(u,q)$ are continuous on sets $\Gamma_{U_T} = \{(y,u) : y \in Y, u \in U_T(y)\}$ and $\Gamma_Q = \{(y,q) : y \in Y, q \in Q(y)\}$, respectively.

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Theorem 2. For the optimality of control $u^0(\cdot)$ and parameter y^0 in problem (3), the existence of $q^0 \in Q(y^0)$ such that

$$\min_{q \in Q(y^0)} \gamma(y^0, q) = \gamma(y^0, q^0)$$

and the fulfillment of the following conditions is necessary and sufficient:

$$\min_{q \in cQ(y^0)} \gamma(y^0, q) = \min_{y \in Y} \min_{q \in Q(y)} \gamma(y,q), \tag{10}$$

$$\min_{v \in V(y^0)} (b(t,v,y^0), \psi(t, y^0, q^0)) = (b(t, u^0(t), y^0), \psi(t, y^0, q^0)) \text{ п.в. на } T. \tag{11}$$

Proof. ***Necessity***. It was shown above that the minimax problem (3) is reduced to the problem of repeated minimization (7). According to the formula (7), this problem can be written using the functional (8) in the following form:

$$\mu(u,y) \to \min,\ u \in U_T(y), y \in Y. \tag{12}$$

Let control $u^0(\cdot)$ and parameter y^0 be optimal in the problem under consideration (3), i.e. in problem (12):

$$\mu(u^0, y^0) = \min_{u \in U_T(y), y \in Y} \mu(u,y).$$

Then, it is clear that

$$\min_{u\in U_T(y^0)} \mu(u,y^0) = \min_{y\in Y}\min_{u\in U_T(y)} \mu(u,y). \tag{13}$$

Since, according to (8) and (9)

$$\min_{u\in U_T(y)} \mu(u,y) = \min_{q\in Q(y)} \gamma(y,q),$$

that ratio (13) can be written as

$$\min_{q\in Q(y^0)} \gamma(y^0,q) = \min_{y\in Y}\min_{q\in Q(y)} \gamma(y,q),$$

that is, equality (10) is true.

Let $q^0 \in Q(y^0)$ be an arbitrary point of the global minimum of the function

$$\eta^0(q) = C(D,\psi(t_0,y^0,q)) + \int_{t_0}^{t_1} C(b(t,u^0(t),y^0),\psi(t,y^0,q))dt,\ q\in Q(y^0).$$

Due to the continuity of the function $\eta^0(q)$ and the compactness of the set $Q(y^0)$, such a point $q^0 \in Q(y^0)$ exists. Then we have:

$$C(D,\psi(t_0,y^0,q^0)) + \int_{t_0}^{t_1} \min_{v\in V(y^0)} C(b(t,v,y^0),\psi(t,y^0,q^0))dt \ge$$

$$\ge \min_{q\in Q(y^0)} [C(D,\psi(t_0,y^0,q)) + \int_{t_0}^{t_1} \min_{v\in V(y^0)} C(b(t,v,y),\psi(t,y^0,q))dt] = \min_{u\in U_T(y^0)} \mu(u,y^0) = \mu(u^0,y^0) =$$

$$= \inf_{q\in Q(y^0)} [C(D,\psi(t_0,y_0,q)) + \int_{t_0}^{t_1} C(b(t,u_0(t),y_0),\psi(t,y_0,q))dt = C(D,\psi(t_0,y_0,q_0)) +$$

$$+ \int_{t_0}^{t_1} C(b(t,u^0(t),y^0),\psi(t,y^0,q^0))dt \ge C(D,\psi(t_0,y^0,q^0)) + \int_{t_0}^{t_1} \min_{v\in V(y^0)} C(b(t,v,y),\psi(t,y^0,q^0))dt\,.$$

It follows from this chain of inequalities that $\gamma(y^0,q^0) = \min_{q\in Q(y^0)} \gamma(y^0,q)$ and

$$\int_{t_0}^{t_1} \min_{v\in V(y^0)} C(b(t,v,y^0),\psi(t,y^0,q^0))dt = \int_{t_0}^{t_1} C(b(t,u^0(t),y^0),\psi(t,y^0,q^0))dt\,.$$

Using the properties of the Lebesgue integral, we obtain the relation (11) from the last equality.

Sufficiently. Let the relations (10) and (11) be fulfilled for some point of the global minimum $q^0 \in Q(y^0)$ of function $q \to \gamma(y^0,q), q\in Q(y^0)$. Then:

$$\mu(u^0,y^0) \le \gamma(y^0,q^0) = \min_{q\in Q(y^0)} \gamma(y^0,q) = \min_{y\in Y}\min_{q\in Q(y)} \gamma(y,l) =$$

$$= \min_{y\in Y}\min_{u\in U_T(y)} \mu(u,y) \le \mu(u,y)\ \forall u\in U_T(y), y\in Y,$$

that is, control $u^0(\cdot)$ and parameter y^0 are optimal in problem (3).

4.Discussion of the results and conclusion.

The paper studies the problem of controlling an ensemble of trajectories of a system formulated as a non-smooth minimax-type control problem. Necessary and sufficient optimality conditions are obtained for this task. They provide a theoretical justification for the method of constructing a solution to problem (3) by solving finite-dimensional problems of the form (10) and (11). The finite-dimensional problem of minimizing function (10) can be solved by mathematical programming methods [3].

The method used to solve the minimax problem (3) can be applied for the case when functional $J(x(\cdot), y)$ has the form

$$J(x(\cdot), y) = \inf_{q\in Q} \max_{m\in M}[(P(y)x(t_1) + m, q),$$

where $P(y)$ - $s\times n$-matrix continuously dependent on $y \in Y$, M and Q are compact subsets of R^s. In this case , the optimality conditions are formulated using the functional

$$\gamma(y,q) = C(D,\psi(t_0, y,q)) + C(M, y) + \int_{t_0}^{t_1} \min_{v\in V(y)} C(b(t,v,y),\psi(t,y,q))dt, \ y \in Y, q \in coQ.$$

Thus, the solution of the non-smooth optimal control problem considered in the paper is reduced to solving finite-dimensional optimization problems. This fact will be useful when developing an algorithm for constructing a solution to the considered control problem.

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